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ON THE STABILITY OF ELASTIC SYSTEMS UNDER RETARDED FOLLOWER FORCES*

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Abstract—The stability of an elastic system with two degrees of freedom (Ziegler's double pendulum) is analyzed under the action of a retarded follower force. The results, obtained via Pontryagin's theorems on roots of exponential polynomials, are presented as regions of stability and instability in the time lag-applied force parameter space. It is shown that in the absence of damping a small time lag will destabilize the system for all values of a compressive force.

INTRODUCTION

THE term "follower force" can be defined as a force of constant magnitude, the direction of which follows the orientation of some part of the associated structure in a prescribed manner. The rocket engine of an aerospace vehicle, for example, is a source of a follower force, since the direction of its thrust is determined by the *simultaneous* orientation of that part of the vehicle to which the engine is attached.

It was first discovered by Ziegler [1] that a follower force, being generally non-conservative, can cause structural instability by flutter, rather than by divergence. He also pointed out an unusual property of follower force systems: velocity-dependent damping, even when vanishingly small, can destabilize the system. A considerable volume of additional literature has appeared on the subject in recent years; a review article has been written by Herrmann [2].

If a force is made to follow the orientation of some part of the structure by means of a servomechanism, then we are dealing with a "retarded follower force". The term *retarded* refers to the inevitable time lag between the input to a directional sensor attached to the structure, and the execution of the appropriate correction to the direction of the force. An example is provided by a simple proportional feedback system used for directional control of missiles, where the thrust gimbal angle θ is determined by the rotation ϕ of the attitude sensor in the following way: $\theta(t) = k\phi(t-\tau)$, τ being the (constant) time lag of the servo-system.

Although the subject of retarded actions has received considerable attention in control theory, its application to mechanical systems has been confined mainly to the linear oscillator. The first paper in this field, by Minorsky [3], showed that an attempt to stabilize a viscously damped pendulum by a servo-operated counterweight can actually result in instability by flutter. The general problem of a linear oscillator and constant time lag has

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since been attacked by several authors, the most recent paper being due to Bhatt and Hsu [4], which also contains a review of the subject.

The effect of a time lag on a mechanical system with multiple degrees of freedom, which includes follower force problems, does not appear to have been studied, despite its potential importance in structural stability of aerospace vehicles. After Beal [5] and Glaser [6] first pointed out that the thrust of existing missiles, if taken as constant, is too small to produce instability (no time lag was considered), attention has been focussed in numerous contractor research reports on the combined action of a follower force with parametric excitation, thrust build-up, and aerodynamic forces.

The present paper will show that a small retardation of a follower force can be destabilizing for all values of the force. The simple model used for the study is, apart from the inclusion of time lag, identical to the double pendulum originally used by Ziegler [1], and later utilized by others [7, 8] in extending Ziegler's work. Although the model bears only a remote relationship to a real structure (a cantilever beam), it was chosen due to its mathematical simplicity, which enabled us to derive the stability criteria in a simple form. Another sacrifice to algebraic brevity was the omission of velocity-dependent damping.

The frequency equation of the system was derived in the conventional manner, resulting in an exponential polynomial possessing an infinite number of roots. The stability criteria were then determined by investigation of the nature of these roots, using certain theorems of Pontryagin [9] on the zeros of transcendental functions.

DERIVATION OF FREQUENCY EQUATION

The double pendulum, shown in Fig. 1, consists of two rigid, weightless rods (AB and BC) of equal lengths l, which carry the masses $2m$ and m at B and C, respectively. Rotation of the rods, measured by the angles ϕ_1 and ϕ_2, is resisted by elastic torsion springs located in the hinges at A and B, the spring constant being c for each hinge.

The load consists of a force of constant magnitude P acting at C. We assume that its direction θ follows the orientation of BC with a constant time lag τ, i.e.

$$\theta(t) = \phi_2(t-\tau). \tag{1}$$

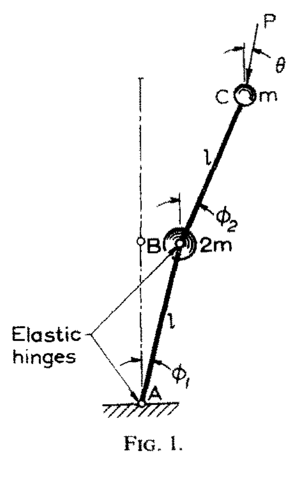

FIG. 1.

As mentioned before, this retarded action could arise if the direction of P were controlled by a servomechanism which obtains its imput from a directional sensor attached to BC.

The linearized equations of motion of the system described are

$$\begin{aligned} 3ml^2\ddot{\phi}_1+ml^2\ddot{\phi}_2+(2c-Pl)\phi_1-c\phi_2+Pl\theta &= 0 \\ ml^2\ddot{\phi}_1+ml^2\ddot{\phi}_2-c\phi_1+(c-Pl)\phi_2+Pl\theta &= 0 \end{aligned} \tag{2}$$

where the dot denotes a time derivative. The solutions of (2) can be expressed in the form

$$\phi_i = A_i \,\mathrm{e}^{zt} \tag{3}$$

or in linear combinations thereof. Equations (1) to (3) lead to the frequency equation

$$\begin{vmatrix} 3ml^2z^2+2c-Pl & ml^2z^2-c+Pl\,\mathrm{e}^{-z\tau} \\ ml^2z^2-c & ml^2z^2+c-Pl+Pl\,\mathrm{e}^{-z\tau} \end{vmatrix} = 0. \tag{4}$$

If we set $\tau = 0$, the left side of (4) becomes a biquadratic polynomial in z; hence the stability of the system can be determined in the usual manner, using the Routh–Hurwitz criterion. It has been shown [1, 7] that instability can occur by flutter only, the critical load being $P_{cr} = 2{\cdot}086\, c/l$.

For positive, nonzero values of τ it is convenient to introduce the dimensionless parameters

$$F = \frac{Pl}{c}, \qquad \Omega = \frac{\tau}{(ml^2/c)^{\frac{1}{2}}}, \qquad \eta = \tau z. \tag{5}$$

Equation (4) then takes the form $H(\eta) = 0$, where

$$H(\eta) = [2\eta^4+(7-4F)\Omega^2\eta^2+(F^2-3F+1)\Omega^4]\mathrm{e}^{\eta}+F[2\eta^2\Omega^2+(3-F)\Omega^4] \tag{6}$$

is an exponential polynomial in η, possessing an infinite number of roots.

The system described can be called stable if ϕ_1 and ϕ_2 remain bounded regardless of the initial conditions, i.e. following any disturbance to the system within the bounds of the linear approximation. Therefore, a sufficient and necessary condition for stability is that all the roots of $H(\eta)$ have negative or zero real parts. In particular, if the roots have negative real parts, then $\phi_1, \phi_2 \to 0$ as $t \to \infty$, and the system is said to be asymptotically stable.

PONTRYAGIN'S STABILITY CRITERIA

Pontryagin's theorems [9] give necessary and sufficient conditions for all the roots of an exponential polynomial to possess negative real parts. Therefore, they are necessary and sufficient conditions of asymptotic stability for linear systems with constant time lag, in a manner similar to Routh–Hurwitz criteria for systems without time lag. The stability criteria of Pontryagin can be stated in several forms, the most convenient of these for our application will now be summarized.

All the roots of the equation $H(\eta) = 0$, where $H(\eta)$ is an exponential polynomial, have negative real parts if, and only if, all of the following conditions are satisfied:

Condition I

Let $H(\eta) = h(\eta, \mathrm{e}^{\eta})$, where h is a polynomial of degree r in η, and of degree s in e^{η}. The polynomial h must possess a principal term, i.e. a term of the form $C\eta^r\,\mathrm{e}^{s\eta}$.

Condition II

Substitute $\eta = iy$, where y is a real variable, and then separate $H(iy)$ into its real and imaginary parts:

$$H(iy) = H_r(y) + iH_i(y).$$

The function $H_i(y)$ must possess exactly $4ks+r$ real zeros in the interval

$$-2k\pi+\varepsilon \le y \le 2k\pi+\varepsilon$$

for a sufficiently large value of the integer k, ε being some constant.

Condition III

For each zero of $H_i(y)$, denoted by a, the inequality

$$H_r(a)H'_i(a) > 0$$

must be satisfied, where the dash denotes a derivative with respect to y.

DERIVATION OF CONDITIONS FOR STABILITY

We will now apply the stability criteria of Pontryagin to the frequency equation (6), with the aim of dividing the $F-\Omega$ coordinate space into stable and unstable regions.

Condition I

In our case $r = 4, s = 1$. Inspection of (6) now reveals the presence of the principal term, namely $2\eta^4\,e^\eta$.

Condition II

Substitution of $\eta = iy$ (y is real) in (6) yields

$$\begin{aligned} H_r(y) &= [2y^4-(7-4F)\Omega^2y^2+(F^2-3F+1)\Omega^4]\cos y+F[-2\Omega^2y^2+(3-F)\Omega^4] \\ H_i(y) &= [2y^4-(7-4F)\Omega^2y^2+(F^2-3F+1)\Omega^4]\sin y. \end{aligned} \tag{7}$$

The function $H_i(y)$ must have $4k+k$ real roots in the interval

$$-2\pi k+\varepsilon \le y \le 2\pi k+\varepsilon .$$

Since $\sin y$ contains $4k$ roots in this interval, Condition II is reduced to the requirement that all the roots of the biquadratic polynomial

$$A(y) = 2y^4-(7-4F)\Omega^2y^2+(F^2-3F+1)\Omega^4 \tag{8}$$

must be real. It is sufficient to confine our attention to the two positive roots only, which are

$$a_{1,2} = \frac{\Omega}{2}[(7-4F) \mp (8F^2-32F+41)^{\frac{1}{2}}]^{\frac{1}{2}}. \tag{9}$$

Real values are obtained only when

$$F^2-3F+1 > 0, \qquad (7-4F)^2-8(F^2-3F+1) > 0 \qquad 7-4F > 0. \tag{10}$$

It is easily verified that all these inequalities are satisfied only if

$$F < \tfrac{1}{2}(3-\sqrt{5}) = 0{\cdot}382 \tag{11}$$

Therefore, (11) represents a necessary condition for asymptotic stability. From now on we will consider only those values of F which satisfy (11).

Condition III

In order to investigate the condition $H_r(a)H_i'(a) > 0$ for all roots a of $H_i(y)$, we must handle separately the following four categories of these roots:

(i) the zeros of $A(y)$
(ii) $a = 0$
(iii) $a = \pm m\pi, m = 1, 3, 5, \dots$
(iv) $a = \pm n\pi, n = 2, 4, 6, \dots$

Category (i)

If a denotes a root of $A(y)$, then we obtain from (7)

$$H_r(a)H_i'(a) = 2F\Omega^2[-2a^2+(3-F)\Omega^2][4a^2-(7-4F)\Omega^2]a\sin a. \tag{12}$$

Since (12) is an even expression in a, no generality is lost by using only the positive roots, given by (9). With the abbreviation

$$K(F) = (8F^2-32F+41)^{\frac{1}{2}} \tag{13}$$

we obtain

$$H_r(a_{1,2})H_i'(a_{1,2}) = F\Omega^6[\pm(2F-1)-K(F)]K(F)a_{1,2}\sin a_{1,2}. \tag{14}$$

The plots of $K(F)$ and $\pm(2F-1)$ in Fig. 2 verify the following inequalities for $F < 0{\cdot}382$:

$$\pm(2F-1)-K(F) < 0.$$

As a consequence, the necessary condition for stability is reduced to $F\sin a_{1,2} < 0$, i.e. there must exist positive, nonzero integers M and N, such that

$$(2M-1)\pi < a_1 < 2M\pi, \qquad (2N-1)\pi < a_2 < 2N\pi \tag{15a}$$

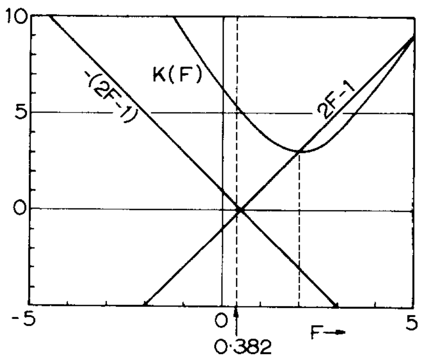

FIG. 2.

if $0 < F < 0{\cdot}382$ (compressive force); or

$$(2M-2)\pi < a_1 < (2M-1)\pi, \qquad (2N-2)\pi < a_2 < (2N-1)\pi \tag{15b}$$

if $F < 0$ (tensile force).

The destabilizing effect of a small time lag for the case of positive F can now be proven. We deduce from (15a) that a necessary condition for stability is $a_1 > \pi$. Since, according to (9), a_1 is proportional to Ω, we can violate the above inequality by making Ω sufficiently small. In fact, we can place a lower bound on the stable range of Ω by showing that the maximum value of a_1 with respect to F is $\frac{1}{2}\Omega(7-\sqrt{41})^{\frac{1}{2}} = 0{\cdot}386\Omega$, occurring at $F = 0$. Therefore, stability can exist only when $\Omega > \pi/0{\cdot}386 = 8{\cdot}53$.

On the other hand, if F is negative, we can always choose a sufficiently small Ω such as to place both a_1 and a_2 in the potentially stable range $(0, \pi)$.

Category (ii)

It is easily verified from (7) that $H_r(0)H_i'(0) = \Omega^8(F^2-3F+1)$. The condition $H_r(0)H_i'(0) > 0$ is thus equivalent to the first inequality in (10), and yields nothing new.

Category (iii)

If $a = \pm m\pi, m = 1, 3, 5, \ldots$, we obtain from (7)

$$H_r(a)H_i'(a) = A(a)B(a) \tag{16}$$

where $A(y)$ is given by (8), and

$$B(y) = 4y^4-(7-6F)\Omega^2y^2+(2F^2-6F+1)\Omega^4. \tag{17}$$

The condition $H_r(a)H_i'(a) > 0$ demands that $A(a)$ and $B(a)$ must have the same sign for all values of a in this category. (Again, no generality is lost by considering the positive values only.)

The roots of $B(y)$ with positive real parts are

$$b_{1,2} = \frac{\Omega}{2}[(7-6F) \mp (20F^2-36F+41)^{\frac{1}{2}}]^{\frac{1}{2}}. \tag{18}$$

We must now investigate two subcases, determined by the value of these roots.

(a) If $2F^2-6F+1 < 0$, i.e. $\frac{1}{2}(3-\sqrt{7}) < F < \frac{1}{2}(3+\sqrt{7})$, only b_2 will be real, and the corresponding behaviour of $B(y)$, together with that of $A(y)$, has been sketched in Fig. 3. It can be deduced that $A(a)$ and $B(a)$ will have the same sign at $a = \pi, 3\pi\ldots$ only when $b_2 < \pi$ and $\pi < a_{1,2} < 2\pi$, where the lower bound on $a_{1,2}$ is determined by (15a). However, it is possible to show that the previously derived necessary condition $\Omega > 8{\cdot}53$ makes a_2 considerably larger than a_1, such that either a_1 or a_2 will always violate the second inequality. We can now conclude that a necessary condition for stability is that both b_1 and b_2 must be real, i.e.

$$F < \tfrac{1}{2}(3-\sqrt{7}) = 0{\cdot}177 \tag{19}$$

which supersedes condition (11) obtained previously.

(b) If (19) holds, the corresponding behavior of $B(y)$, also shown in Fig. 3, leads us to the

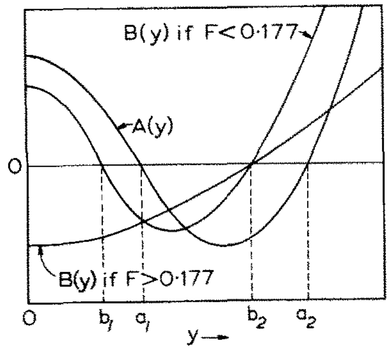

FIG. 3.

conclusion that $A(a)\,B(a) > 0$, $a = \pi, 3\pi \ldots$ only if

$$(2M-1)\pi < b_1 < (2M+1)\pi, \qquad (2N-1)\pi < b_2 < (2N+1)\pi \tag{20a}$$

if F is positive, or

$$(2M-3)\pi < b_1 < (2M-1)\pi, \qquad (2N-3)\pi < b_2 < (2N-1)\pi \tag{20b}$$

if F is negative. The integers M and N are determined by (15a) or (15b).

Category (iv)

With $a = \pm n\pi$, $n = 2, 4, 6, \ldots$, equations (7) yield

$$H_r(a)H_i'(a) = A(a)C(a) \tag{21}$$

where

$$C(y) = 2y^4-(7-2F)\Omega^2 y^2+\Omega^4 \tag{22}$$

and $A(y)$ is again defined in (8). The zeros of $C(y)$ are always real for $F < 0{\cdot}177$, the positive roots being

$$c_{1,2} = \frac{\Omega}{2}[(7-2F) \mp (4F^2-28F+41)^{\frac{1}{2}}]^{\frac{1}{2}}. \tag{23}$$

Since the behaviour of $C(y)$ is similar to that of $B(y)$ in the same range of F, the condition $H_r(a)H_i'(a) > 0$, $a = 2\pi, 4\pi \ldots$, becomes

$$(2M-2)\pi < c_1 < 2M\pi, \qquad (2M-2)\pi < c_2 < 2N\pi \tag{24}$$

for positive as well as negative values of F. The values of M and N are again defined by (15a) or (15b).

The conditions (15), (19), (20) and (24) are necessary and sufficient for the system to be asymptotically stable. If the values of the parameters F and Ω are such that any one of the above conditions is violated, asymptotic stability will not exist.

DISCUSSION OF RESULTS

It was previously noted that Pontryagin's criteria represent conditions for asymptotic stability. The possibility of steady-state solutions, which are also considered stable motions, should be investigated separately. However, it is unlikely that additional zones of stability will be found in this manner. Previous work on follower forces, as well as on linear oscillators with a time lag, has shown that steady-state solutions occupy lines, rather than finite regions, in the parameter space of the system (such as the $F-\Omega$ coordinate plane). Consequently, they are of no practical interest.

The most important result of the analysis is the observation, discussed in the previous section, that a sufficiently small time lag destabilizes the system for all positive (compressive) values of F. Even under the most favourable time lag, the upper bound of the critical load was shown to be $F = 0{\cdot}177$, which is considerably smaller than the corresponding value for the same system without the retarded action, namely $F = 2{\cdot}086$.

Another noteworthy feature of the analysis is the apparent paradox that the case of vanishing time lag (letting $\Omega \to 0$ in Pontryagin's stability criteria) does not yield the same results as the case of zero time lag (setting $\tau = 0$ in the original frequency equation). Part of the reason for this can be traced back to the transformation of equation (4) into (6). During this procedure both sides of the equation were multiplied by Ω^4, thus making (6) invalid for $\Omega = 0$. Nevertheless, we can argue that the results we obtained are valid for all nonzero, positive values of Ω, even if they are very small. Consequently, a transition from instability to stability will take place at $\Omega = 0$, $0 < F < 2{\cdot}086$, as $\Omega \to 0$.

A similar situation arises when the influence of linear–viscous damping is investigated: in a certain range of F a transition from instability to stability is observed as the damping coefficient is reduced from a vanishingly small value to zero. A physical insight into this transition has been proposed by Herrmann and Jong [8]. By considering the actual motion of the pendulum, they showed that the "degree of stability" (rate of amplitude increase) undergoes a continuous change as the damping coefficient is reduced to zero. We suggest that a similar technique could be used, at least in principle, in explaining the effect of vanishing time lag on the stability of the systems.

The behaviour of the system is radically different if F is negative. Since all the roots of $A(y)$, $B(y)$ and $C(y)$ are proportional to Ω, a sufficiently small time lag can place all the positive roots in the stable region $(0, \pi)$ regardless of the value of F.

A more detailed investigation of stability of the system is ideally suited for digital computer application. We used equally spaced values of F and Ω as the input, each set of values of (F, Ω) producing an output in the form of the character "U" if any of the stability conditions were violated, or the character "S" if all the conditions were satisfied. The printout was arranged as a two-dimensional array of these characters in the F–Ω coordinate system. A sample printout, covering a large range of time-lag, is shown in Fig. 4.

Additional information about the structure of stable zones can be obtained by plotting a_1/Ω, b_1/Ω etc. as functions of F, as in Fig. 5. For positive F we see that $b_1 < a_1 < c_1$, $b_2 < a_2 < c_2$, while these inequalities are reversed for negative F. The stability criteria (15), (20) and (24) can then be replaced by the simpler conditions

$$b_1 > (2M-1)\pi, c_1 < 2M\pi$$

$$b_2 > (2N-1)\pi, c_2 < 2N\pi$$

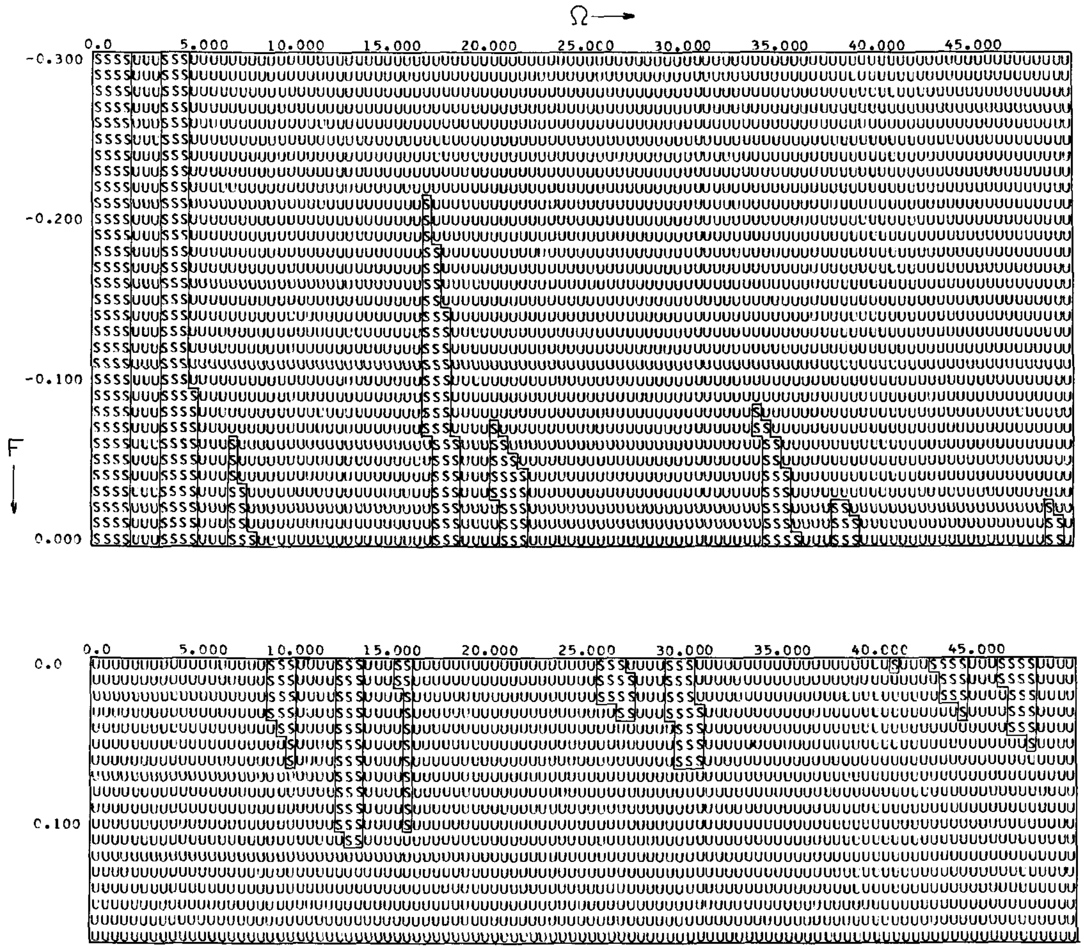
Ω
0.0
5.000
10.000
15.000
20.000
25.000
30.000
35.000
40.000
45.000
-0.300
-0.200
-0.100
0.000
F
0.0
0.100

FIG. 4.

if $F > 0$, and

$$b_1 < (2M-1)\pi, c_1 > (2M-2)\pi$$

$$b_2 < (2N-1)\pi, c_2 > (2N-2)\pi$$

if $F < 0$, where M and N are nonzero, positive integers. Therefore, the boundaries between stable and unstable zones can be readily determined by plotting the curves

$$b_{1,2}(F,\Omega) = (2N-1)\pi, \qquad c_{1,2}(F,\Omega) = (2N-2)\pi, \qquad N = 1, 2\ldots$$

which has been done in Fig. 6. The stable zones have the form of strips, almost parallel to the F-axis, bounded by the lines $c_2 = (2N-2)\pi$ and $b_2 = (2N-1)\pi$, and truncated in their length by lines $c_1 = (2N-2)\pi$ and $b_1 = (2N-1)\pi$, which cut "diagonally" across the F-Ω plane.

The absence of velocity-dependent damping from present analysis deserves a special mention. Its introduction would complicate the analysis considerably, and it is almost certain that both necessary and sufficient conditions for stability cannot be derived in a simple form. It would nevertheless be interesting to discover whether a small damping coefficient would introduce additional stable zones, particularly in the region of small Ω and positive F, or if it would merely change the boundaries of the present stable zones.

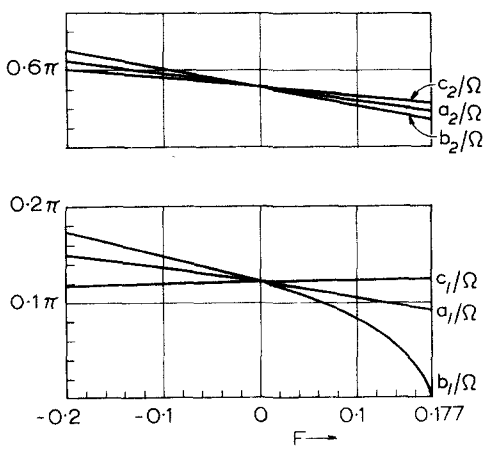

FIG. 5.

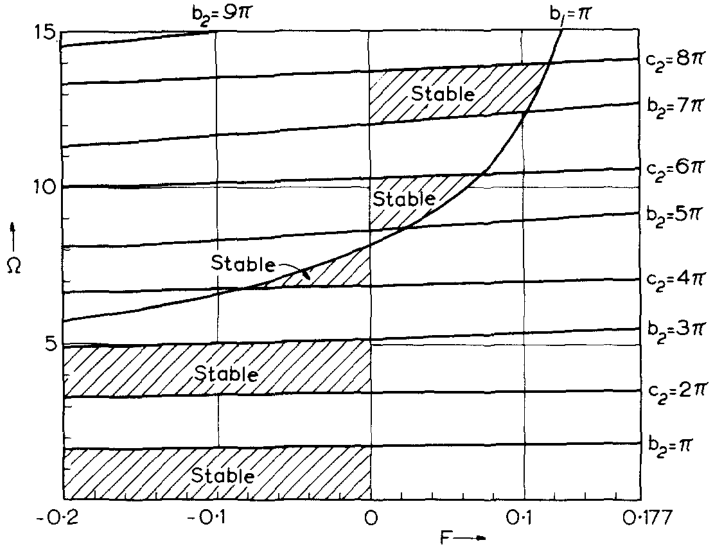

FIG. 6.

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Абстракт—Дается анализ устойчивости упругих систем, обладающих двумя степенями свободы /двойный маятник Циглера/, под влиянием запаздывающей следящей силы. Результаты, полученные на основе теорем Понтрягина, касающихся корней зкспотенциальных полиномов, представляются как области устойчивости и неустойчивости в параметрическом пространстве времени запаздывания и приложенной силы. Окагввается, что при отсуствии демпфирования малое время запаздывения будет причиной неустойчивости системы для всех значений сжимаемой силы.